B.Sc. Part III 5th Paper

METRIC SPACES (CONTINUED)

Q: Let (X, d) be a metric space. Let

$$d^*(x,y) = \min\{1, d(x,y)\}.$$

Prove that d^* is a metric for X.

Soln.

We shall show that

(a) $d^*(x,y) \geq 0$

(b) $d^*(x,y) = 0 \Leftrightarrow x = y$

(c) $d^*(x,y) = d^*(y,x)$

(d) $d^*(x,y) \leq d^*(x,z) + d^*(z,y)$.

(a) Given that $d^*(x,y) = \min\{1, d(x,y)\}$

i.e. $d^*(x,y) = 1$ or $d(x,y)$ which ever is minimum.

But since d is a metric.

$\Rightarrow d(x,y) \geq 0$

$\Rightarrow d^*(x,y) = \min\{1, \geq 0\}$

$\Rightarrow d^*(x,y) \geq 0$. (a) proved

Proof of (b)

Let $d^*(x,y) = 0 \Leftrightarrow \min\{1, d(x,y)\} = 0$

$\Leftrightarrow d(x,y) = 0$

$\Leftrightarrow x = y$ [as d is metric and $d(x,y) = 0 \Leftrightarrow x = y$

So, $d^*(x,y) = 0 \iff x = y$ (b) Proved

Proof of (c) we have to show that

$$d^*(x,y) = d^*(y,x)$$

Now $d^*(x,y) = \min\{1, d(x,y)\}$

$= \min\{1, d(y,x)\}$

because d is a metric $\Rightarrow d(x,y) = d(y,x)$

$\Rightarrow d^*(x,y) = d^*(y,x)$ (c) Proved

Proof of (d) we show that

$$d^*(x,y) \leq d^*(x,z) + d^*(z,y) \quad \text{--- (1)}$$

Given that $d^*(x,y) = \min\{1, d(x,y)\}$.

$\Rightarrow d^*(x,y) \leq 1$ so that if any of the terms on RHS of eq(1) is unity, we will have the result proved.

If $d^*(x,z) < 1$ and $d^*(z,y) < 1$

$\Rightarrow d^*(x,z) = \cancel{d(x,z)} \min\{1, d(x,z)\} = d(x,z)$

and $d^*(z,y) = d(z,y)$.

$\therefore d^*(x,z) + d^*(z,y) = d(x,z) + d(z,y)$

$\geq d(x,y)$ [$\because d$ is a metric]

$\geq \min\{1, d(x,y)\}$

$\geq d^*(x,y)$

$\Rightarrow d^*(x,y) \leq d^*(x,z) + d^*(z,y)$ (d) Proved